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Heat transfer characteristics of a boundary-layer flow driven by a power-law shear over a semi-infinite flat plate

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Abstract

The hydrodynamic and thermal boundary layer similarity flows driven past a semi-infinite impermeable flat plate by a power-law shear with asymptotic velocity profile $U_\infty(y) = \beta y^{-1/2}$ $(y \to \infty, \beta > 0)$ is considered (y denotes the coordinate normal to the plate). Assuming that the buoyancy and viscous dissipation effects may be neglected, the special cases of an isothermal and of an adiabatic flat plate are examined both analytically and numerically.

1. Introduction

The shear driven flows, like the wall-driven Couette-flow, the wind-driven Ekman-flow, the Lock-type flows near to the interface of two parallel streams etc. belong to the classical topics of fluid mechanics. Due to their wide technical and environmental applications, the general research interest in the shear driven flows is still present in our days. Recently, the adjustment of a *zero pressure gradient* laminar flow near a flat impermeable boundary to an exterior power-law velocity profile of the form

$$U_\infty(y) = \beta y^\alpha \quad (y \to \infty, \beta > 0) \tag{1}$$

has been investigated for a wide range of values of the exponent α by Weidman et al. [1]. The aim of the present paper is to examine the heat transfer characteristics of the boundary layer flow past an impermeable semi-infinite flat plate corresponding to the case $\alpha = -1/2$ by assuming that the buoyancy and viscous dissipation effects are negligible. We shall assume that the plate temperature $T_w(x)$ varies as a power γ of the distance x along the plate, and will present exact analytical solutions for the isothermal $(\gamma = 0)$ and the adiabatic $(\gamma = -1/3)$ cases, respectively, for arbitrary values of the Prandtl number. To our best knowledge this problem has not been considered before.

2. Basic equations

Assuming that the buoyancy and viscous dissipation effects may be neglected, the laminar boundary layer equations of a viscous and incompressible fluid describing a *zero pressure gradient* flow and the corresponding energy equation can be written in non-dimensional form as

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = \frac{\partial^3\psi}{\partial y^3} \tag{2}$$

$$\frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y} = \frac{1}{Pr}\frac{\partial^2 T}{\partial y^2} \tag{3}$$

where x and y are the non-dimensional Cartesian coordinates measured along the plate and normal to it, respectively, T is the non-dimensional temperature and ψ is the non-dimensional stream function, which is defined in the usual way as $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$ with u and v being the non-dimensional velocity components along x and y axes, respectively. Eqs. (2) and (3) are

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solved subject to the following impermeability, non-slip and thermal boundary conditions

$$\psi(x,0)=0,\quad \frac{\partial\psi}{\partial y}(x,0)=0\quad\text{and}\quad \frac{\partial\psi}{\partial y}\to\beta y^{\alpha}\quad (y\to\infty)$$
$$T(x,y)=T_{\rm w}(x)\quad\text{and}\quad T\to 0\quad (y\to\infty) \tag{4}$$

Weidman et al. [1] have shown that by the similarity transformation

$$\psi(x,y)=x^{(\alpha+1)/(\alpha+2)}\cdot f(\eta),\qquad \eta=x^{-1/(\alpha+2)}\cdot y \tag{5}$$

Eq. (2) reduces to the ordinary differential equation

$$(\alpha+2)f'''+(\alpha+1)ff''-\alpha f'^2=0 \tag{6}$$

along with the boundary conditions

$$f(0)=0,\quad f'(0)=0,\quad f'(\eta)\to\beta\eta^{\alpha}\quad\text{as } \eta\to\infty \tag{7}$$

where primes denote differentiation with respect to η. In the exactly solvable cases $\alpha=-1/2$ and $\alpha=-2/3$, the effect of a lateral suction and injection of the fluid has been considered recently by Magyari et al. [2].

The aim of the present paper is to examine the heat transfer characteristics of the boundary layer flow past an impermeable semi-infinite flat plate corresponding to the case $\alpha=-1/2$. In this case, we assume that the wall temperature distribution has the form

$$T_{\rm w}(x)=T_\infty+T_0x^{\gamma} \tag{8}$$

where T_∞ is the ambient temperature of the surrounding fluid, and $T_0>T_\infty$ and γ are constants. Under this assumption the energy equation (3) admits the similarity solutions of the form

$$T(x,y)=T_\infty+T_0\cdot x^{\gamma}\cdot\vartheta(\eta) \tag{9}$$

where the dimensionless temperature ϑ satisfies the ordinary differential equation

$$(\alpha+2)\vartheta''+Pr\cdot[(\alpha+1)f\vartheta'-\gamma(\alpha+2)f'\vartheta]=0 \tag{10}$$

along with the boundary conditions

$$\vartheta(0)=1,\quad \vartheta(\infty)=0 \tag{11}$$

where Pr is the Prandtl number. Thus, the solution (9) corresponds to the wall temperature distribution (8). However, we are concerned here mainly with the special cases of an isothermal ($\gamma=0$) and of an adiabatic ($\vartheta'(0)\equiv 0$) plate, respectively.

3. Solution and discussion

The solution of the flow problem (6) and (7) for $\alpha=-1/2$ has been given for an impermeable plate by Weidman et al. [1] and for a permeable plate by Magyari et al. [2]. In the former case it reads:

$$f(\eta)=(24\beta^2)^{1/3}\frac{\sqrt{3}\cdot Ai'(z)+Bi'(z)}{\sqrt{3}\cdot Ai(z)+Bi(z)} \tag{12}$$

$$f'(\eta)=\frac{2\beta^2}{3}\eta-\frac{1}{6}f^2 \tag{13}$$

where $Ai(z)$ and $Bi(z)$ are the Airy functions (see e.g. [3]) and $z=(\beta/3)^{2/3}\cdot\eta$.

Further, it is easy to show that in the case of an isothermal flat plate ($\gamma=0$), the solution of the energy equation (10) can be given by quadratures as follows:

$$\vartheta(\eta)\equiv\vartheta_{\rm is}(\eta)=1-\frac{\int_0^z[\sqrt{3}\cdot Ai(t)+Bi(t)]^{-2Pr}\,{\rm d}t}{\int_0^\infty[\sqrt{3}\cdot Ai(t)+Bi(t)]^{-2Pr}\,{\rm d}t} \tag{14}$$

Thus, the heat transfer coefficient $h\equiv-\vartheta'_{\rm is}(0)$ is obtained as

$$h=\left(\frac{\beta}{3}\right)^{2/3}\frac{[\sqrt{3}\cdot Ai(0)+Bi(0)]^{-2Pr}}{\int_0^\infty[\sqrt{3}\cdot Ai(t)+Bi(t)]^{-2Pr}\,{\rm d}t} \tag{15}$$

where $\sqrt{3}\cdot Ai(0)+Bi(0)=2\cdot 3^{-1/6}/\Gamma(2/3)$ (see [3]).

The dependence of h on the Prandtl number Pr is plotted (for $\beta=1$) in Fig. 1 and in Fig. 2 a couple of temperature profiles $\vartheta_{\rm is}(\eta)$ corresponding to different values of Pr are shown. As expected, the heat transfer coefficient is always positive and increases with increasing value of the Prandtl number.

Now, we turn our attention to the case of an adiabatic plate, that is, $\vartheta'(0)\equiv 0$. Thus, integrating the energy equation (10) once and taking into account the thermal boundary conditions (11) one obtains (for $\alpha=-1/2$):

$$\vartheta'(0)=Pr\cdot\int_0^\infty\left(\frac{1}{3}f\vartheta'-\gamma f'\vartheta\right){\rm d}\eta \tag{16}$$

After a partial integration, the requirement $\vartheta'(0)\equiv 0$ leads to the equation

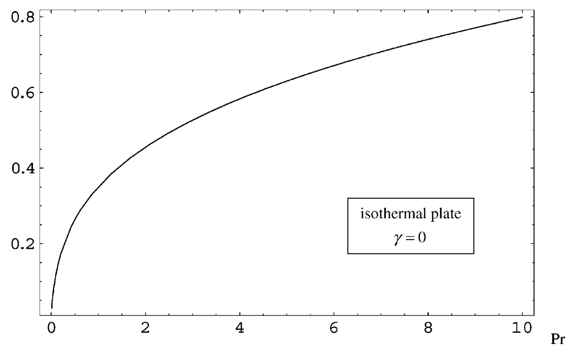

Fig. 1. Dependence of the heat transfer coefficient (15) on the Prandtl number Pr in the isothermal case (for $\beta=1$).

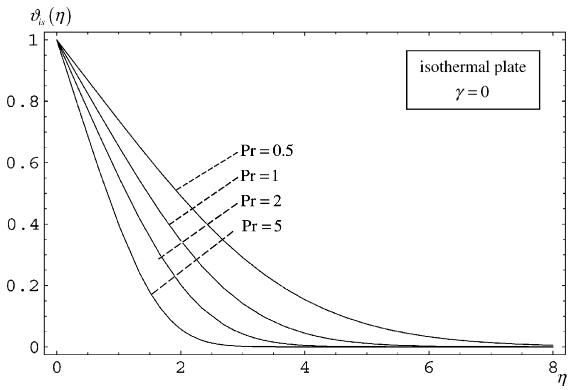

Fig. 2. Temperature profiles (14) plotted against η for different Prandtl numbers in the isothermal case (for $\beta = 1$).

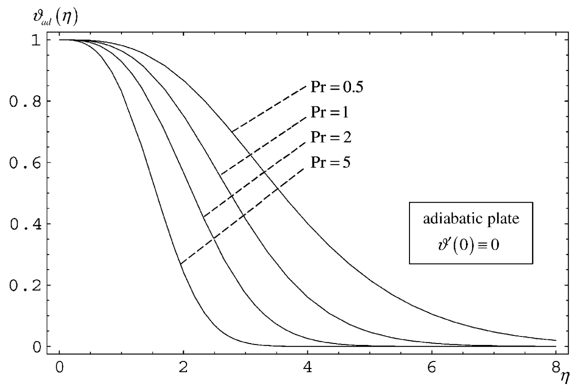

Fig. 3. Temperature profiles (20) plotted against η for different Prandtl numbers in the adiabatic case (for $\beta = 1$).

$$\lim_{\eta\to\infty}[f(\eta)\vartheta(\eta)] = (1+3\gamma)\cdot\int_0^\infty f'\vartheta\,d\eta \tag{17}$$

where, according to the third equation (7), $f(\eta) \to 2\beta\eta^{1/2}$ as $\eta \to \infty$. Now, assuming that $\vartheta(\eta)$ goes to zero as $\eta \to \infty$ faster than $\eta^{-1/2}$ (see also below), the left-hand side of Eq. (17) vanishes and we thus conclude that the adiabatic case is realized for the wall temperature distribution of the form

$$T_w(x) = T_\infty + T_0 \cdot x^{-1/3} \tag{18}$$

which corresponds to the temperature distribution in a plume (see [4], p. 151). The corresponding dimensionless temperature field is then immediately obtained as

$$\vartheta(\eta) = \exp\left(-\frac{1}{3}Pr\cdot\int_0^\eta f(t)\,dt\right) \tag{19}$$

Having in mind Eq. (12), we further obtain the explicit solution

$$\vartheta(\eta) \equiv \vartheta_{ad}(\eta) = \left[\frac{\sqrt{3}\cdot Ai(0)+Bi(0)}{\sqrt{3}\cdot Ai(z)+Bi(z)}\right]^{2Pr} \tag{20}$$

corresponding to the identically vanishing wall temperature gradient $\vartheta'_{ad}(0) = 0$. Taking into account the asymptotic behavior of the Airy functions (see [3]), we easily obtain the asymptotic expression of the temperature field (20) in the form:

$$\vartheta(\eta) \to \left[\frac{2(\beta/3)^{1/3}\sqrt{\pi}}{\Gamma(2/3)}\eta^{1/4}\exp\left(-\frac{2\beta}{9}\eta^{3/2}\right)\right]^{2Pr} \quad \text{as } \eta\to\infty \tag{21}$$

Thus, the left-hand side of Eq. (17) actually vanishes as assumed above.

As an illustration, a couple of temperature profiles (20) corresponding to different values of the Prandtl number

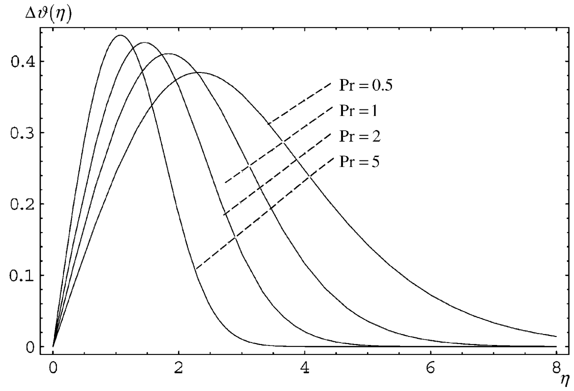

Fig. 4. Excess temperature profiles (22) plotted against η for different Prandtl numbers (for $\beta = 1$).

Pr are shown in Fig. 3. It is seen that for the same value of *Pr*, the temperature of the fluid over the adiabatic plate, $\vartheta_{ad}(\eta)$, as expected, is for any $0 < \eta < \infty$ higher than $\vartheta_{is}(\eta)$ over its isothermal counterpart. This circumstance is explicitly seen in Fig. 4, where the excess temperature

$$\Delta\vartheta(\eta) \equiv \vartheta_{ad}(\eta) - \vartheta_{is}(\eta) \tag{22}$$

is shown for different values of *Pr*. It can be seen from this figure that the maximum of $\Delta\vartheta$ increases with increasing value of *Pr* and is shifted to smaller values of η.

4. Summary and conclusions

Heat transfer in the forced convection flow past an impermeable flat plate in outer shear flow $U_\infty(y) = \beta y^{-1/2}$ as $y \to \infty$, $\beta > 0$ has been considered in the isothermal and adiabatic case, respectively. The analytical solutions obtained show that the peak of the excess temperature $\vartheta_{adiabatic}(\eta) - \vartheta_{isotherm}(\eta)$ increases with the increasing value of the Prandtl number *Pr* and, at the same time, it is shifted to smaller values of the similarity variable $\eta = y/x^{2/3}$.

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